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Numerical results on a hypercubic cell spin glass model

G Parisi†‡, F Ritort§‖¶ and J M Rubí§¶

† Dipartimento di Fisica, Università di Roma II, 'Tor Vergata', via E Carnevale, I-00173 Roma, Italy

‡ Istituto Nazionale di Fisica Nucleare Sezione di Roma, Italy

§ Departament de Física Fonamental, Universitat de Barcelona, Diagonal 648, 08028 Barcelona, Spain

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Abstract. We study a spin glass hypercubic cell in D dimensions (i e. hypercubic lattice with $L=2$ where L is the lattice size) for different values of D by means of Monte Carlo simulation In the limit $D\to\infty$ this model converges to the SK model as well as for hypercubic lattices. We confirm the usual interpretation of the spin glass phase Since our model is more similar to hypercubic lattices than to the infinite-range model these results suggest that broken replica symmetry is useful to study finite-dimensional hypercubic lattices above a certain critical dimension.

1. Introduction

Since the SK model was first solved in 1975 [1] and the instability found afterwards [2] there have been several attempts to find the correct solution. Among them there has been one proposal [3] which nowadays seems to be correct and has been one of the main sources of new results in spin glasses and optimization problems.

Up to now the problem of testing the correctness of this solution has not been theoretically and numerically feasible. In the first case, the crucial assumption of ultrametricity has not been fully tested. Although the solution is locally stable [4], this does not necessarily imply that this solution is the correct one, because there is no available classification of all possible solutions of the saddlepoint equations in replica space, so there is no proof that there are other solutions to the replica equations which better describe the system. Numerical tests supporting the correctness of this solution are very important; unfortunately, the enormous amount of time needed to make Monte Carlo simulations for systems with a number of spins greater than 500 makes them unfeasible [5].

It would be of great interest to look for finite-range models which converge to the mean-field solution and entail less amount of computation time than the SK model.

Hypercubic lattices in finite dimensions are good candidates to this end. Concretely, an Ising model in a hypercubic lattice converges to mean field when the dimension goes to infinity [6]. The same happens in spin glasses Working with a hypercubic lattice has the inconvenience that the lattice size L and the dimensionality D are two parameters to control in the numerical simulation.

‖ On leave at Dipartimento di Fisica, Università di Roma II 'Tor Vergata', via E Carnevale, I-00173 Roma, Italy

¶ Bitnet D3FFFRF2@EB0UB011.

We propose a new model which we think behaves similarly to the SK model and does not have the inconvenience of the usual lattice. It is a hypercubic cell (i.e. a hypercubic lattice with lattice size $L=2$) of dimension D with connectivity to its D nearest neighbours and free boundaries.

For ordered systems it can be proven that the Ising model in a hypercubic cell topology also goes to mean-field theory when the dimension goes to infinity. We feel that the same will happen in the spin glass case.

On the other hand, expansions in inverse powers of dimensionality for the free energy in ordered systems show that the hypercubic cell (in dimension $2D$) and lattice (in dimension D) resemble each other closely at finite dimensions. For this reason, the results in the hypercubic cell in finite dimensions will show us if the SK model is really pathological [7] or, on the other hand, if the same features appear in finite-dimensional lattices [8, 19].

From the theoretical point of view we expect the thermodynamics of this model to coincide with that of the usual SK model, and to differ only by terms which disappear when $N\to\infty$. In this respect this model differs by these terms defined on a random graph of fixed connectivity z [20], where the thermodynamics in the $N\to\infty$ limit is a function of z, which can be controlled theoretically only for asymptotically large z using an expansion in $1/z$.

If we want to compare our system with the Bethe lattice we have to take account of the fact that its behaviour is strongly dependent on the specific boundary conditions [25]. Concretely, the Bethe lattice with correlated boundary conditions shows the same behaviour as the random lattice [26], where there is replica symmetry breaking and a lot of thermodynamic states. For uncorrelated boundary conditions (for example, fixing randomly the spins in the tips of the tree) the behaviour of the Bethe lattice changes completely and the replica-symmetric solution becomes stable [27]. Since in both cases small loops are rare (in the first case they occur with probability $O(1/N)$ and in the second case there are no loops), we think that the first specific boundary conditions will better reproduce the properties of the hypercubic cell (where there are small loops). Since in this case the Bethe lattice behaves as the random graph, we think it suffices to compare our results with this last model and, concretely, the case of fixed connectivity rather than that of average connectivity since it reproduces our specific topology with more fidelity.

The system has 2^D spins (each with D nearest neighbours) and is described by an Ising Hamiltonian of the usual type,

$$H=-\sum_{(ij)} J_{ij}\sigma_i\sigma_j$$

and probability of couplings,

$$P(J_{ij})=\tfrac{1}{2}\delta(J_{ij}-J)+\tfrac{1}{2}\delta(J_{ij}+J)$$

where we have taken $J=1/\sqrt{D}$ with $\overline{J_{ij}}=0$, $\overline{J_{ij}^2}=1/D$ to normalize extensive magnitudes ($\overline{(\ldots)}$ denotes average over samples)

For this system we have taken as the order parameter the overlap probability distribution $P(q)$ defined in the SK model by

$$P(q)=\frac{\mathrm{d}x(q)}{\mathrm{d}q}$$

where $q(x)$ is the order parameter function. To find $P(q)$ numerically in the SK model it is useful to consider certain properties of correlation functions at zero external field.

One evaluates

$$P(q)=\overline{P_J(q)} \tag{1}$$

with

$$P_J(q)=\langle\delta(q-q_{12})\rangle_2^J$$

where $\langle\dots\rangle_2^J$ denotes the thermal average over two systems 1 and 2 with identical couplings J_{ij}. The superscript J recalls the fact that this average is strongly dependent on coupling realization.

The parameter q_{12} is calculated in the SK model in two different ways:

(i) In the first case, the fact that $\langle\sigma_1\sigma_2\dots\sigma_k\rangle^2=\int_0^1 q^k(x)\,\mathrm{d}x$ allows us to write

$$q_{12}=\frac{1}{N}\sum_{i=1}^{N}\sigma_i^1\sigma_i^2 \tag{2}$$

(ii) In the second case, from Callen's identity,

$$\langle\sigma_i\rangle=\left\langle\tanh\beta\left(\sum_{\langle j\,nn\,i\rangle}J_{ij}\sigma_j\right)\right\rangle$$

one obtains

$$q_{12}=\frac{1}{N}\sum_{i=1}^{N}m_i^1m_i^2 \tag{3}$$

with

$$m_i=\tanh\beta\left(\sum_{\langle j\,nn\,i\rangle}J_{ij}\sigma_j\right).$$

This work can be split into two main parts: (i) equilibration of the hypercube system and (ii) study of equilibrium properties in different dimensions.

In the equilibration we have started with expression (2) for $P(q)$. In the study of equilibrium properties we have used expression (3). In any case, we have discovered that (2) and (3) yield the same results.

The use of expression (3) to study equilibrium properties has the advantage of avoiding discretization for q_{12} which appears in (2) when the size of the system is small (less than approximately 1000 spins). In this way, a continuous and soft $P(q)$ without irregularities can be obtained.

2. Equilibration

One of the most important problems in disordered systems with strong metastability is the great amount of numerical time needed to reach equilibrium at a certain temperature starting from random configurations. To discover the minimal number t_0 of Monte Carlo steps needed to equilibrate samples we have calculated $P(q)$ from expression (2) following a useful procedure proposed in earlier works [9]

We make a system evolve during a number t_0 of Monte Carlo steps from an arbitrary initial configuration. Then we memorize the configuration of the spin $\{\sigma_i(t_0)\}$. After that, we let the system evolve during a number, for instance t_0, of Monte Carlo steps to lose memory of the earlier configuration.

Next, during a period of time, which for the sake of convenience we have again chosen to be t_0, we evaluate the auto-overlap from expression (2),

$$q_{12}^{a}(t, t_0) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i(t_0)\sigma_i(t_0+t) \qquad t_0 \leqslant t \leqslant 2t_0$$

and the probability distribution

$$P_J^{a}(q) = \frac{1}{t_0} \sum_{t=t_0}^{2t_0} \delta(q - q_{12}^{a}(t, t_0)) \tag{4}$$

from which we obtain $P^{a}(q) = \overline{P_J^{a}(q)}$. Altogether, for each t_0 we have considered, each evolution has taken a number of Monte Carlo steps approximately equal to $3t_0$.

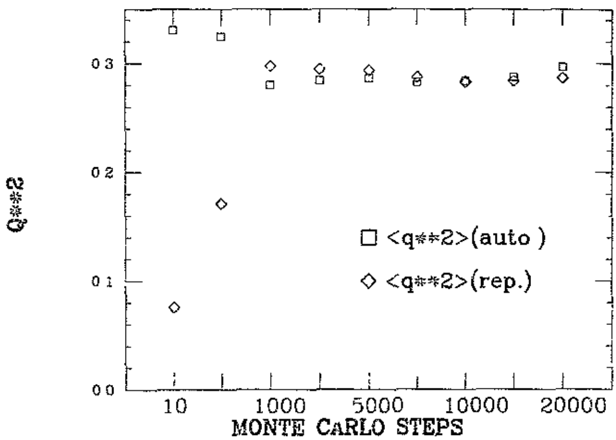

Figure 1. Equilibration time of the second moment from (6) against Monte Carlo time t_0 for $D=8$

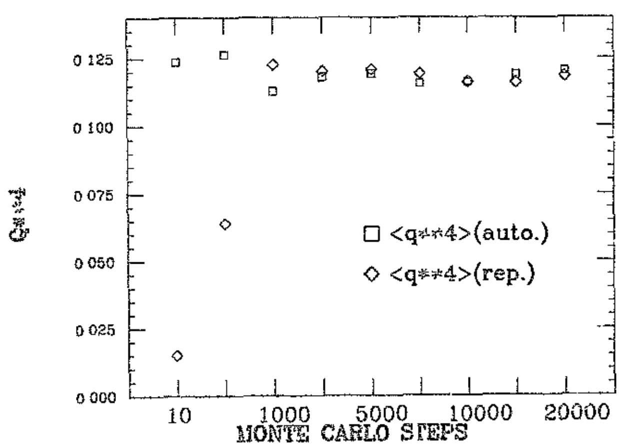

Figure 2. Equilibration time of the fourth moment from (6) against Monte Carlo time t_0 for $D=8$.

In the same process of equilibration we find the overlap probability distribution during the last t_0 Monte Carlo steps between two replicas wholly independent but identical in the coupling realization (identical samples):

$$q_{12}^{r}(t)=\frac{1}{N}\sum_{i=1}^{N}\sigma_i^1(t)\sigma_i^2(t)$$

with

$$P_J^r(q)=\frac{1}{t_0}\sum_{t=2t_0}^{3t_0}\delta(q-q_{12}^r(t)). \tag{5}$$

When the two probability distributions coincide, we have reached the equilibrium. In consequence, their moments coincide:

$$\langle q^k\rangle_a=\int_0^1 q^k P^a(q)\,\mathrm{d}q \tag{6}$$

$$\langle q^k\rangle_r=\int_0^1 q^k P^r(q)\,\mathrm{d}q. \tag{7}$$

We have studied the equilibration at $T=0.5$ for dimensions 6, 8 and 10, corresponding to 64, 256 and 1024 spins. For $D=6$ we have studied 100 samples, 80 for $D=8$ and 40 for $D=10$. The number of samples being small, it was useful to estimate the minimal equilibration time.

In figures 1 and 2 we show the equilibration of moments q^2 and q^4 as functions of the number t_0 of Monte Carlo steps for the case $D=8$.

In figure 3 we show how $P(q)$ calculated from (5) evolves towards equilibrium for the case $D=10$. The tendency is very similar (as we have seen in $N=256$ or $D=8$) to results obtained in the SK model shown by Bhatt and Young [10].

For the case $D=12$ the strong metastability makes the former method unfeasible. Therefore, we have used a simulated annealing procedure [11] to reach the equilibrium. In this way we have reduced the computation time by a factor of greater than 10.

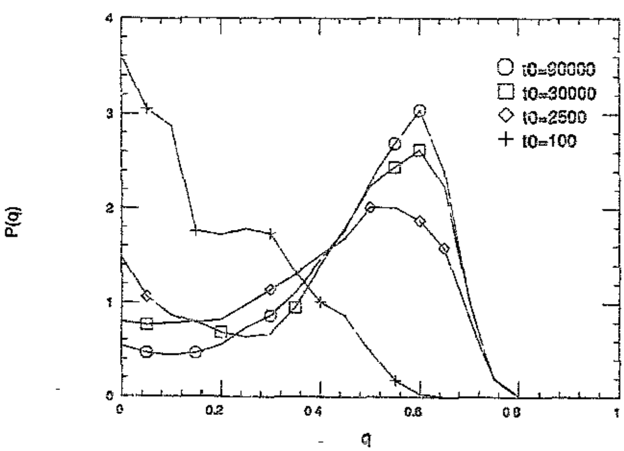

Figure 3. Equilibration of $P(q)$ for $N=1024$ ($D=10$). The average is performed over 40 samples. The symbols are guides to the eye.

3. Equilibrium properties

Once we have reached equilibrium we calculate $P(q)$ originating from expression (3). To obtain good statistics we evolved eight identical replicas in parallel and in each Monte Carlo step we calculated the 28 possible overlaps.

3.1. On the transition temperature

We expect that the hypercube cell (in the thermodynamic limit) has the transition at $T=1$ (as in the SK model). In figure 4 we plot $P(q)$ for $D=6$, 8, 10 and 12, and in the inset we show the scaling of the standard deviation with $N^{-1/2}$, which goes to zero as $N\to\infty$. These results show clearly that $q\simeq 0$ at $T=1$. The value of the internal energy is compatible with the theoretical prediction $U=-0.5$. This proves that the transition temperature is not greater than 1.

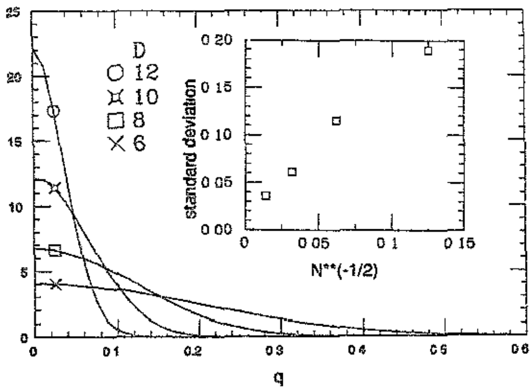

Figure 4. $P(q)$ obtained using expression (3) for $D=6$, 8, 10 and 12 at $T=1$. The inset shows the standard deviation of $P(q)$ plotted against $N^{-1/2}$ It goes to zero as $N\to\infty$

3.2. Order parameter function

We have studied the hypercube at $T=0.5$ for the cases $D=5$, 6, 7, 8, 10 and 12 corresponding to 32, 64, 128, 256, 1024 and 4096 spins with 200, 200, 160, 160, 60 and 20 samples, respectively. For large sizes we considered a small number of samples, since for each such case the computational effort is considerable. For large sizes and few samples non-self-averaging quantities experience sample-to-sample fluctuations which give large error bars, but for self-averaging ones these are considerably reduced (even if statistical errors exist for each sample, the main source of errors always comes from averaging over the samples). In figure 5 we show $P(q)$ for five different cases together with the usual theoretical result for the SK model solving the replica equations at infinite order of replica symmetry breaking [12]. The most remarkable fact seen in figure 5 is that, for $D\geqslant 10$, q_{max} (defined as the position of q where $P(q)$ is a maximum) seems to fall below the theoretical result for the SK model.

As we will explain later we attribute this fact to finite-dimensionality effects. For the case $D=12$ large error bars are attributed to the small number of samples. We have had to compromise in CPU time between a good sampling and a good equilibration.

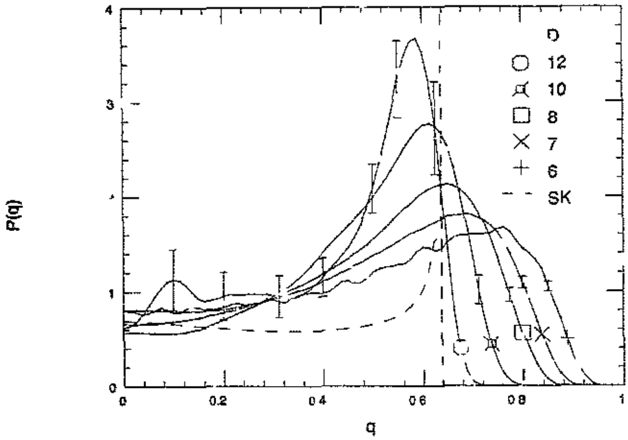

Figure 5. $P(q)$ obtained using expression (3) for the cases $D=6, 7, 8, 10$ and 12 at $T=0.5$ We also show the theoretical prediction in the SK model with $q_{max}=0.6375$ extracted from [12]). Error bars are shown for $D=12$ and for lower dimensions the typical size for them is also plotted.

For lower dimensions, the error bars are smaller (especially for $D=6$–8) and typical ones for each dimension are shown in figure 5. As can be seen, the point $q=0$ is not independent of the size of the sample. It seems that $P(q)$ has at $q=0.29$ a point which is independent of the size of the sample, at least for $D\leqslant 10$. For $D=12$ this fact cannot be confirmed due to large error bars. We do not know any reason for this fact but we think that this effect will disappear as we go to higher dimensions.

We have to note that in our model we have an interplay between finite-size and finite-connectivity effects. Our interpretation of the results is based mainly on the assumption that the behaviour of the system can be split into two regimes, between which there is a crossover We have not arrived at the size for which this crossover is seen but we can predict it to occur over several thousands of spins. In the first regime finite-size effects completely mask finite-dimensionality ones. In contrast, in the second regime, finite-dimensionality effects mainly determine the evolution of the system towards the thermodynamic limit. In other words, the system begins to be more sensitive to dimensionality effects than to finite-size ones.

In order to study this assumption, let us begin our analysis, checking if the following relations, satisfied in the SK model [13], are satisfied by our numerical results:

$$\chi=\beta\left(1-\int_0^1 q(x)\,\mathrm{d}x\right)=1 \tag{8}$$

$$U=-\frac{\beta}{2}\left(1-\int_0^1 q^2(x)\,\mathrm{d}x\right). \tag{9}$$

In table 1 we show the results of different dimensions studied in order to test (8). We do not see a convergence towards the expected result. To the contrary, an increasing discrepancy appears. Since relation (9) does not suffer from finite-size effects at infinite dimensions (when we recover the SK model [23]) we can consider deviations from this formula to be attributed more to finite-dimensionality effects than to finite-size ones. In figure 6 we show the numerical values of the energy for the hypercube plotted against $N^{-1/2}$. By doing a least squares linear fit to the data (the straight line in the

Table 1. Results of different dimensions studied in order to test (8)

Dimension	Susceptibility
5	0 8813 ± 0.02
6	1 0026 ± 0 018
7	1 037 ± 0.02
8	1 055 ± 0 02
10	1.072 ± 0 03
12	1 163 ± 0 026

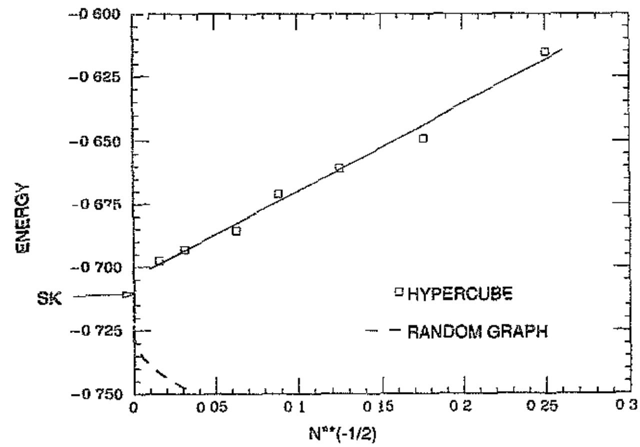

Figure 6. Energy $U = -\Sigma_{(i,j)} J_{ij}\sigma_i\sigma_j$ for different sizes For all cases the error bars are smaller than the size of the square symbols We can extrapolate $U = -0\,705 \pm 0\,005$ in approximate agreement with the value for the SK model $U = -0.710$ The broken line is the theoretical prediction for the random graph with fixed connectivity z putting $z = D$.

figure), we can extrapolate $U = -0.705 \pm 0.005$, in approximate agreement with the theoretical prediction for the SK model at first order of replica symmetry breaking $U = -0.710$ [24] (for $T = 0.5$, which is a temperature not too low, the results obtained for self-averaging quantities at first order of replica symmetry breaking and infinite order are nearly the same; for the energy we expect a difference less than 10^{-3}, which is clearly indistinguishable in our numerical simulations). The data for the numerical energy show a linear behaviour when plotted against $N^{-1/2}$, but it is not clear if the convergence is good. It is probable that in the regime of sizes we have studied, and particularly for the energy, finite-size effects are completely dominating the convergence.

As we have said in the introduction we hope that the thermodynamics of our system will finally recover that of the random graph (first introduced by Viana and Bray [22] for the case of average connectivity). Also in figure 6 we show the theoretical prediction obtained from recent results in the random graph [20] with fixed connectivity z, analysed by means of the $1/z$ expansion, using $z = D$. In this paper it is verified that in the limit $z \to \infty$ the random graph coincides with the SK model and the $1/z$ corrections at finite temperature can be computed (the discrepancy between the result for q_1 at $T = 0$ in the limit $z \to \infty$ between [20] and the result in [24] are due to an extrapolation error in [24]). As we can see, when the size and dimension of the system increase, the energy decreases. We expected that the numerical data for the energy would progressively reach the theoretical prediction shown in figure 6 by the broken line. This is not the case and it could be that the effective convergence radius for the expansion is rather small or the $N^{-1/2}$ corrections are so strong as to mask the effect.

In figure 7 we plot the order parameter $\langle q^2\rangle$ against $N^{-1/2}$ together with the predicted result in the SK model $\langle q^2\rangle \simeq 0.2895$. The main comment to be made now is that we see a good linear behaviour in this plot, newly characteristic of finite-size effects, but the results do not show any tendency to converge to the correct one. A crossover behaviour is expected to correct this bad convergence when the interplay between finite-size and finite-dimensionality effects begin to be dominated by the latter.

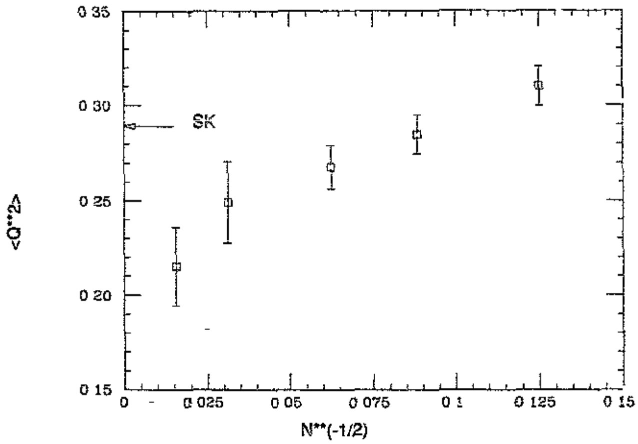

Figure 7. Order parameter $\langle q^2\rangle$ against $N^{-1/2}$ From the correct energy $U \simeq -0.710$ and relation (9) we would expect it should converge towards $\simeq 0.2895$.

Now, we return to the question of how to separate finite-dimensionality effects from finite-size ones. As we do not have at our disposal a complete theory generally agreed on spin glasses, we have to proceed on the basis of suppositions and heuristic arguments. Since relation (9) is independent of the size of the system at infinite dimensions, one can regard deviations from this formula to be attributed more to finite-dimensionality effects than to finite-size ones. Introducing finite-dimensionality corrections to this expression one can write (for $T = 0.5$)

$$-U + \int_0^1 q^2(x)\,\mathrm{d}x - 1 \simeq \frac{a}{D} + \frac{b}{D^2} + \mathrm{O}\left(\frac{1}{D^3}\right). \qquad (10)$$

In figure 8 we plot our numerical results for several dimensions of the system. We do not see any tendency of our results to converge towards the result at infinite dimensionality (the point at the origin of the axes). This means that the expansion (10) has a contribution from high-order terms, being indicative of something stated before, i.e. the effective radius of convergence for expansions in powers of $1/D$, at least for this model of spin glass, could be very small.

Considering the behaviour of the energy, susceptibility and $\langle q^2\rangle$, together with the evolution of $P(q)$, as we increase the size of the system, our interpretation of the numerical results is as follows. It is probable that hypercubic lattices at finite dimensions over a critical one will share some of the novel features of the infinite-range model.

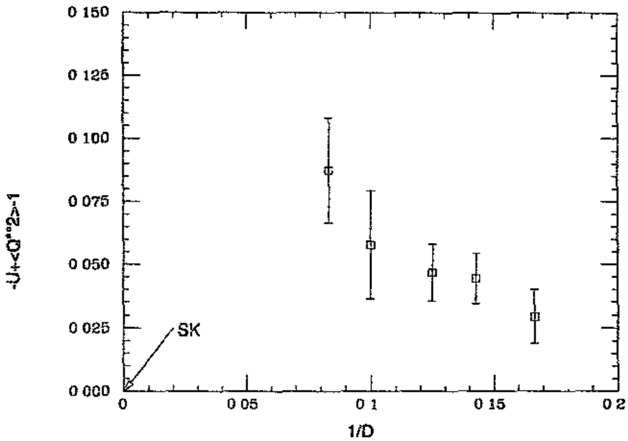

Figure 8. Bray and Moore's relation (10) satisfied exactly for the SK model (the point at the origin of the axes) It suggests that high-order finite-dimensionality corrections to this formula could be very important for the hypercubic cell.

It seems as if the interplay between finite-size and finite-dimensionality effects can be separated in two regimes between which there is a crossover regime.

In the first regime when the size of the system grows, finite-size effects are dominant. The energy and $\langle q^2 \rangle$ show a behaviour characteristic of finite-size effects, i.e. they decrease as $N^{-1/2}$. The behaviour of the $P(q)$ function is very interesting in this regime. When the size of the system increases, the value of q_{max} decreases, as seen in the SK model. Comparing our result, for example $P(q)$ at dimension $D=8$, with those shown in the lattice at $D=4$ [19] (to have the same connectivity as the cell) we discover a surprising similarity. We also find coincidence with results shown for the SK model [10] but to a lesser degree. It looks as if finite-size effects are very strong in this range, leaving aside secondary questions regarding the topology of the lattice. Numerical results would be very welcome for the behaviour of the magnitudes shown in this work for the case of the random graph already mentioned with the same connectivity as the hypercubic cell to test to what extent this affirmation is true.

The second regime where finite-dimensionality effects become important appears when N reaches several thousands of spins. It can be predicted clearly by looking at the $P(q)$ function at the precise moment at which the position of q_{max} falls below the position calculated for the SK model. This causes a decrease in all integrals of the type $\int_0^1 q^k(x)\,dx$, explaining why the susceptibility and $\langle q^2 \rangle$ seem to converge too far from the correct result. The form of the $P(q)$ is now no longer similar to that of the SK model but the similarity to that of the hypercubic lattice with half dimension remains. The crossover will appear when, at a certain size, the first appearance of the tendency for q_{max} to decrease when the size increases is inverted and q_{max} begins to increase with the dimensionality of the cell. This means that the maximum overlap q_{max} for hypercubic lattices at finite dimensions is always smaller than the corresponding value for the SK model and converges to it as the dimension of the lattice increases. This expected crossover can surely only be predicted but not seen in the simulations because we need to go to higher dimensions. In fact, an increase in the dimension of the system of one unit doubles the size of the system and the amount of computational time grows enormously. This crossover is predictable too (but less clearly due to numerical errors) from the behaviour of the energy, susceptibility and $\langle q^2 \rangle$. As the linear plots of $N^{-1/2}$ against, for example, the energy and $\langle q^2 \rangle$ do not converge to the correct result when $N \to \infty$, there will be a moment at which the tendency has to vary substantially to finally recover the results expected for infinite dimension.

These numerical results seem to us very difficult to explain in terms only of finite-size effects. Since the main features of our results so far explained have not been found in the SK model, we think that finite-dimensionality effects and the close connection of our system with hypercubic lattices are truly important for their explanation.

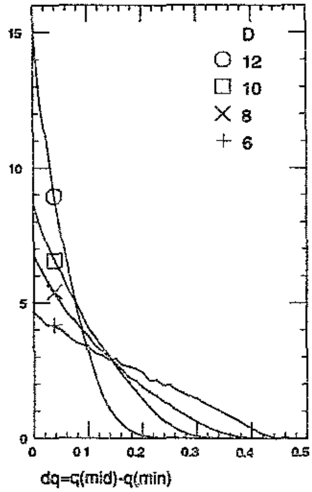

Figure 9. Ultrametricity test in the hypercubic cell We plot the probability distribution of the difference between the two smallest overlaps when the maximum lies between 0.4 and 0 5

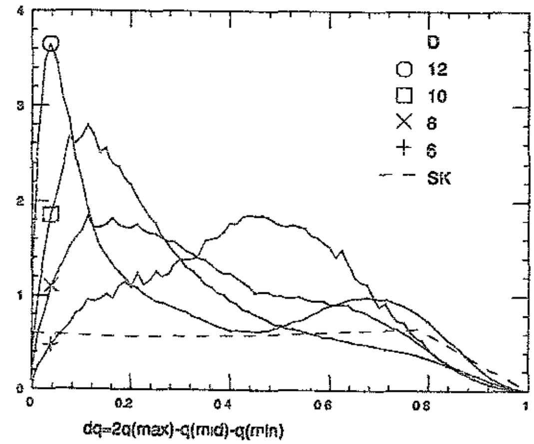

Figure 10. Probability distribution of triangles in the phase space of pure states for the hypercubic cell when the maximum overlap lies between 0 4 and 0.5. The SK model predicts a delta function at $\Delta q = 0$ of weight 0.5 plus a continuous part (broken line). The peak shows that the probability of equilateral triangles is finite.

It could be that there is an intricate interplay between finite connectivity and finite-size effects, which is outside the scope of our discussion. Anyway, the interpretation given above seems to us to be the simplest we can give. Let us remember that not only do we lack a theory generally agreed on finite-dimensional spin glasses but in the 'simplest' case, i.e. the SK model, a complete understanding of finite-size effects is still lacking.

Finally, a simple remark regarding the similarity between our system and hypercubic lattices would be in order. One could think on the possibility of a difference between the two systems as we consider the cell for odd dimensions because in this case the spins are less likely to flip freely than in the lattice. This fact is not important since we are working at $T=0.5$ and it is a high enough temperature for this effect to be smeared by thermal fluctuations. At low temperatures (let us say less than ≈ 0.1) we should see large oscillations in the energy plotted against $N^{-1/2}$ as we pass from an odd value of dimensionality to an even value (as is seen in the SK model with binary couplings at low temperatures when N changes from even to odd values and exact statistical mechanics for small samples is investigated [21]).

If this interpretation of the results is correct and broken replica symmetry is a useful concept for hypercubic lattices at finite dimensions, the phase space organization of pure states should not show any crossover as is the case for $P(q)$. In fact, this is what we expect if the replica symmetry-breaking scheme did not depend on the dimensionality of the lattice and this is what seems to be the case.

3.3. *Ultrametricity*

We have investigated ultrametricity in the hypercube of sizes $D=6$, 8, 10 and 12 for 20 samples in each case. The results are shown in figures 9 and 10. Like former studies [15], we confirm the physical description of the spin glass phase with broken replica symmetry in this case.

In figure 9 for each triad of the eight systems we evaluated the difference between the minimum and the middle overlap when the maximum lies between 0.4 and 0.5. Our results show strong evidence of ultrametricity.

In figure 10 we plot the probability distribution of the value $\Delta q = 2q_{max} - q_{min} - q_{mid}$ when the maximum overlap lies between 0.4 and 0.5. Also, we show the theoretical prediction [16] for the SK model (a delta function at $\Delta q = 0$ plus a continuous term). Our results are in good agreement with the theoretical prediction.

4. Conclusion

We have studied a hypercube cell at $T=0.5$ in the low-temperature phase for dimensions $D=6$, 7, 8, 10 and 12. The motivation has been two-fold:

(i) Since mean field is found when the dimensionality of the hypercubic lattice goes to infinity, we expect that the same will happen for the hypercubic cell. A simulation of this model will allow us to test the main features of the spin glass phase in the SK model.

(ii) Following the known fact that in ordered systems the hypercubic cell (in two dimensions) is more similar to the hypercubic lattice (in dimension D) than to the infinite-range model, we want to explore if the novel features found in the SK model are also shared by hypercubic lattices over a certain critical dimension [17].

As always happens in frustrated systems, strong metastability has made equilibration painful (especially in the case $D = 12$).

In our system there is an interplay between dimensionality and finite-size effects. Even if we do not have a clear way to separate both effects, we have found that the simplest interpretation of the results allows us to differentiate two regimes where a different behaviour is expected.

The first regime is dominated by finite-size effects The $P(q)$ shows the typical behaviour of the hierarchical solution in the SK model. The thermodynamical quantities also show a linear behaviour when plotted against $N^{-1/2}$ as in the SK model.

The second regime, predicted from the fact that all the magnitudes converge towards incorrect results, should appear when the size exceeds several thousands of spins (we cannot predict exactly when it can be seen). In this regime the system would be more sensitive to finite-dimensionality effects than to finite-size ones.

Results in figure 8 cast doubt on the utility of $1/D$ expansions in this model of spin glass. Comparing our results with those obtained in the random graph model with fixed connectivity z using an expansion in $1/z$ we do not find any agreement because of $O(N^{-1/2})$ effects or possibly large $1/D^2$ corrections.

By studying the organization of pure states in phase space we have found ultrametricity and we have given support to the usual physical interpretation of broken replica symmetry [18]. This is a very important result because the ultrametricity structure is the key assumption in the theory of spin glasses.

All these results lead us to confirm that the usual physical interpretation of the SK model is correct, and to the suspicion that the novel features of the spin glass phase for the SK model could also be a reality for hypercubic lattices at finite dimensions over a critical value.

We hope that our numerical results will serve as a guide for future theories in this very controversial subject.

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